C. Homotopy and CW-complexes

Homotopy groups are very powerful, especially for CW-complexes

Recall if A is a topological space

$$f : \coprod_{i \in I} S^{n-1} \to A \quad \text{a continuous map}$$

then

$$X = A \cup_f \Big(\coprod_{i \in I} D^n\Big) = \frac{A \amalg \big(\coprod_{i \in I} D^n\big)}{\sim}$$

where $x \in \partial\big(\coprod D^n\big)$ is identified with $f(x) \in A$

is said to be the result of attaching n-cells to A

a relative CW complex is a pair (X, A) s.t.

1) X is a topological space
2) A is a closed subspace

and $\exists$ a sequence of subspaces $X^{(n)}$, $n = -1, 0, 1, \ldots$ called the n-skeleta s.t.

3) $X^{(-1)} = A$
4) $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells
5) $X = \bigcup_{i=-1}^{\infty} X^{(i)}$ and
6) $B \subset X$ is closed $\iff$ $B \cap X^{(n)}$ closed

for all n

if there is an n such that $X = X^n$ then X is <u>n-dimensional</u>, otherwise X is <u>∞-dimensional</u>

a <u>CW complex</u> is a space X s.t. $(X, \emptyset)$ is a relative CW pair

<u>Remark</u>: If X has a finite number of cells we don't need 6).

6) is the W in CW complex, it means use the "weak" topology on X

If X is a CW complex a subcomplex $A \subset X$ is a closed subset that is a union of cells in X

(X, A) is called a <u>CW pair</u>

<u>exercise</u>: if (X, A) a CW pair then X/A is a CW complex

<u>examples</u>:
1) a 1-dimensional CW complex is a graph
2) Surfaces are CW complexes
<u>exercise</u>: show this
as are all manifolds
3) if X, Y are CW complexes then so is $X \times Y$
<u>exercise</u>: work out the CW structure

a map $f: X \to Y$ between CW complexes is cellular if $f(X^{(n)}) \subset Y^{(n)}$ for all n

Cellular Approximation Thm:

If $f: X \to Y$ is a continuous map between CW complexes and f is cellular on $A \subset X$ a sub CW complex then f is homotopic, rel A, to a map $g: X \to Y$ s.t. g is cellular

You can find a proof in many books, e.g. Hatcher.

we can now compute some homotopy groups

$$\boxed{\pi_k(S^n) = 0 \text{ for } k < n}$$

Proof: given $f: (S^k, s_0) \to (S^n, x_0)$ (s_0, x_0 the 0-skeleta)

can homotop (fixing image of s_0) to a map g such that $g(S^k) \subset (k\text{-skeleton of } S^n) = \{x_0\}$

$\therefore g$ constant

We compute $\pi_n(S^n)$ later

What about $\pi_k(S^n)$ for $k > n$?

this is very hard in general!

example: $\pi_3(S^2) \neq 0$

to see this let $f: S^3 \to S^2$ be the Hopf map recall: $S^3 \subset \mathbb{C}^2$ unit sphere

$S^1 \subset \mathbb{C}$ unit sphere act on S^3

S^3/S^1 = complex lines in $\mathbb{C}^2$

$= \mathbb{C}P^1 \cong S^2$ exercise

$f: S^3 \to S^3/S^1 = S^2$ is the quotient map

also $\mathbb{C}P^2 = \mathbb{C}P^1 \cup_f B^4$ exercise

i.e. attach 4-cell to S^2

if $f \simeq 0$ then $\mathbb{C}P^2 \simeq S^2 \vee S^4$

easy to see $[S^2] \in H^2(S^2 \vee S^4)$

has $[S^2] \cup [S^2] = [0] \in H^4(S^2 \vee S^4)$

but Poincaré duality says

$[S^2] \in H^2(\mathbb{C}P^2)$ has

$[S^2] \cup [S^2] \neq 0$ in $H^4(\mathbb{C}P^2)$

$\therefore f \not\simeq 0$ and $\pi_3(S^2) \neq 0$

later we will see $\pi_3(S^2) \cong \mathbb{Z}$

lemma 19:

X a CW complex then the inclusion map $i: X^{(n)} \to X$ induces an isomorphism

$$i_*: \pi_k(X^{(n)}) \to \pi_k(X) \quad \text{for } k < n$$

and a surjection

$$i_*: \pi_n(X^{(n)}) \to \pi_n(X)$$

Proof: i_* is surjective for $k \le n$ by argument above (about $\pi_k(S^n) = 0$ for $k < n$)

now if $k < n$ we show injective

given $f: S^k \to X$ and $g: S^k \to X$ in $\pi_k(X)$

we can assume their image is in $X^{(k)}$ from above

if $[f] = [g]$ in $\pi_k(X)$ then they are homotopic

by $H: S^k \times [0,1] \longrightarrow X$ (fixing s_0)

$S^k \times [0,1]$ has a CW structure of dim $k+1$

by the Cellular Approximation Th$^{\underline{m}}$ we can

homotop H, staying fixed on $S^k \times \{0,1\}$ and

on $\{s_0\} \times [0,1]$ to $G: S^k \times [0,1] \longrightarrow X^{(k+1)} \subset X^{(n)}$

so $[f] = [g]$ in $\pi_k(X^{(n)})$

lemma 20:

$$\boxed{\text{If } (X,A) \text{ a relative CW complex and } A \text{ is contractible then } X/A \simeq X}$$

we prove this later but now we see some consequences

call a space X k-connected if

$$\pi_\ell(X) = 0 \quad \forall\, \ell \le k$$

Thm 21:

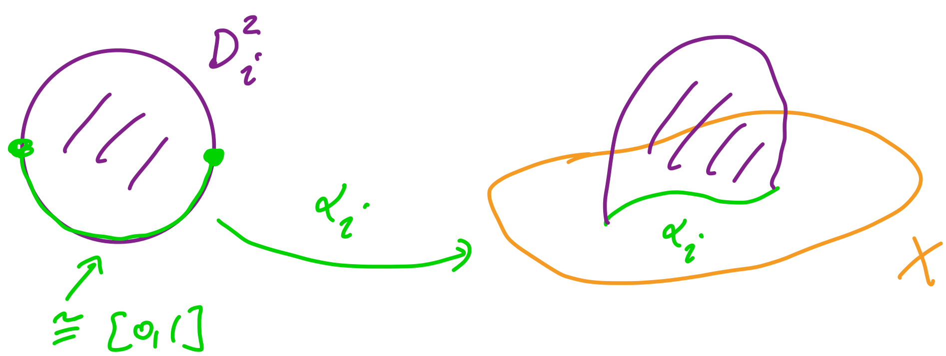
> If X is a k-connected CW complex then $X \simeq X'$ where X' is a CW complex containing a single vertex and no cells of dimension 1 through k

Proof: let e_0 be a vertex of X

since X is 0-connected it contains paths $\alpha_1 \dots \alpha_n : [0,1] \to X$ connecting e_0 to the other verticies $e_1 \dots e_n$

by cellular approximation we can assume $\alpha_i : [0,1] \to X^{(1)}$

for each i we can glue a 2-disk D_i^2 to X as follows:

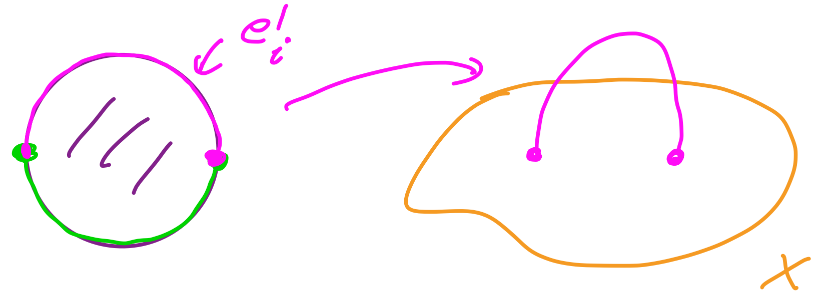


let X_1' be the resulting space

note: X_1' a CW complex

you get X_1' by adding

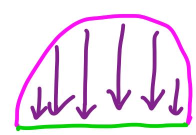
the 1-cell



then the 2-cell

$e^2_i = D^2_i$ along $\alpha_2 \cup e^1_i$

Clearly $X_1' \simeq X$ since we can deformation retract X_1 to X



note: $\cup e^1_i$ is a contractible subcomplex of X_1'

so if we set $X_1 = X_1' / \cup e^1_i$

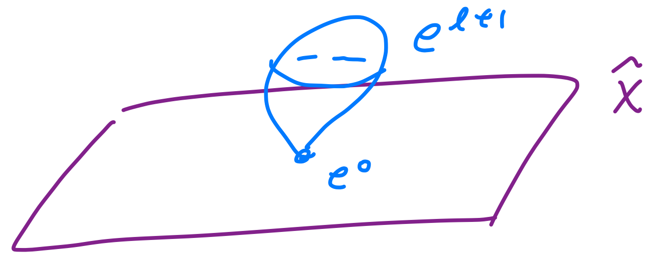
lemma 20 says $X_1 \simeq X_1' \simeq X$

and X_1 is a CW complex with one vertex

Now assume $X \simeq \hat{X}$ where $\hat{X}$ a CW complex with one vertex and no cells of dim $1, \dots, l$ for $l < k$

we want to find $\hat{X}'$ s.t. $\hat{X}' \simeq X$ and no cells of dim $1, \dots, l+1$

each $l+1$ cell e^{l+1} is attached by a map $\partial e^{l+1} \xrightarrow{f} X^{(l)} = \{e_0\}$ that is constant



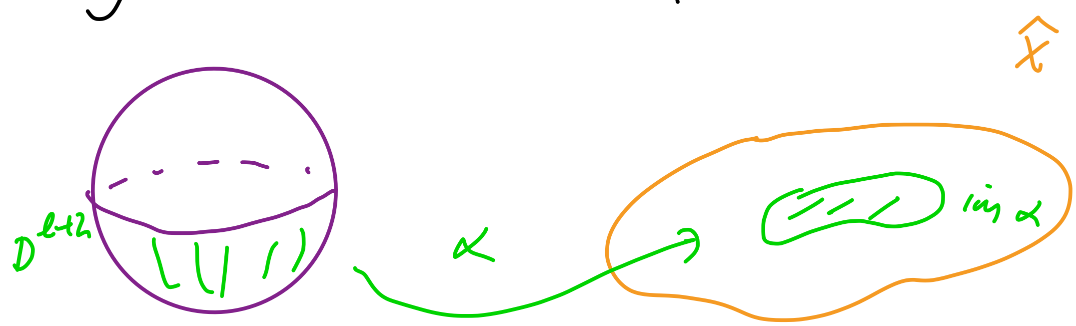
so e^{l+1} gives an element of $\pi_{l+1}(\hat{X}) = 0$

so $\exists$ a disk $\alpha : D^{l+2} \to \hat{X}$ such that

$\alpha(\partial D^{l+2}) = e^{l+1}$ and we can

assume $\alpha(D^{l+2}) \subset \hat{X}^{(l+2)}$ (cellular approx)

now glue D^{l+3} to $\hat{X}$ by



to get $\tilde{X}$ note $\tilde{X}$ has a new $(l+2)$ cell e

and a new $(l+3)$ cell e'

now as above D^{l+3} can be contracted to D^{l+2}

so $\tilde{X} \simeq \hat{X}$

e is a contractible subcomplex of $\tilde{X}$

so $\hat{X}' = \tilde{X}/e \simeq \tilde{X} \simeq \hat{X}$

and $\tilde{X}$ has one less $(l+1)$ cell than $\hat{X}$

continuing we can get $\hat{X}'$ with no $(l+1)$ cells. ▦

Corollary 22:

> If X is a CW-complex and $\pi_i(X) = 0 \ \forall i$ then X is contractible!

Proof: if X a finite dimensional complex then Th^{m} 21 says it is $\simeq \{pt\}$

if X is infinite dimensional then X having weak topology allows us to conclude some thing ▦

Corollary 23:

> if X is an n-connected CW-complex then $\tilde{H}_k(X) = 0 \ \ \forall k \leq n$
>
> i.e. $\pi_k(X) = 0 \ \forall k \leq n \Rightarrow \tilde{H}_k(X) = 0 \ \forall k \leq n$

recall $\tilde{H}_k$ is reduced homology

$\tilde{H}_k = H_k$ for $k > 0$

$H_0 = \tilde{H}_0 \oplus \mathbb{Z}$

Proof: use Cellular homology

recall the chain groups are

$C_k(X) =$ free abelian group generated by k-cells

$$= \bigoplus_{\#\, k\text{-cells}} \mathbb{Z}$$

so if X is n-connected, Th^{m} 21 says

$X \simeq \hat{X}$ with $\hat{X}^{(n)} = \{e_0\}$

so $C_k(\hat{X}) = 0 \quad \forall k = 1, \ldots, n$

$C_0(\hat{X}) = \mathbb{Z}$ so $H_0(\hat{X}) = \mathbb{Z}$ but $\tilde{H}_0(\hat{X}) = 0$

$\therefore H_k(\hat{X}) = 0 \quad \forall k = 1, \ldots, n$ too $\boxplus$

Th^{m} 24:

$$\boxed{\text{If } (X,A) \text{ a CW pair and } \pi_n(X,A) = 0\ \forall n \text{ then } X \text{ deformation retracts to } A \text{ (i.e. } X \simeq A)}$$

Proof: just like proof of 21 and 22

exercise: give proof $\boxplus$

Th^m 25 (Whitehead's Theorem):

If X, Y are CW complexs with base points $x_0 \subset X^{(0)}$ and $y_0 \in Y^{(0)}$ with Y connected, and $f: (X, x_0) \to (Y, y_0)$ is a map such that

$$f_*: \pi_k(X, x_0) \to \pi_k(Y, y_0) \text{ is isom. } \forall k$$

then $f: X \to Y$ is a homotopy equivalence

Remarks:

1) f satisfying the hypothosis is called a weak homotopy equivalence

so th^m says: "for CW complexes a weak htpy equiv is a htpy equiv."

2) Two spaces can have isomorphic π_n $\forall n$ and not be homotopy equivalent. You really need a map between the spaces to induce the isomorphism.

for example, let $X = \mathbb{R}P^2 \times S^3$ and $Y = S^2 \times \mathbb{R}P^3$

from Alg Top I we know $\pi_1(X) \cong \mathbb{Z}/_2 \cong \pi_1(Y)$

and since $S^2 \times S^3$ covers both X and Y

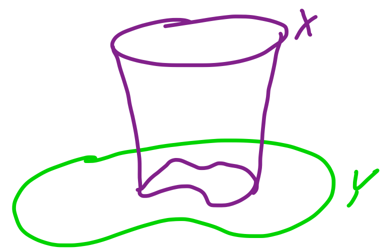
lemma 18 says

$$\pi_n(X) \cong \pi_n(S^2 \times S^3) \cong \pi_n(Y) \quad \forall n \geq 2$$

but X and Y are not weakly htpy equiv since $H_5(X) = 0$ since X is not orient. and $H_5(Y) \cong \mathbb{Z}$ since Y is orient.

Proof: given $f: X \to Y$ we can make it cellular and consider the mapping cylinder

$$C_f = (X \times [0,1]) \cup Y \big/ {(x,0) \sim f(x)}$$



exercise: C_f has a CW structure where $X \times \{1\}$ is a subcomplex

recall $C_f \simeq Y$ infact $j: C_f \to Y : \begin{array}{l} (x,t) \longmapsto f(x) \\ y \longmapsto y \end{array}$ is the homotopy inverse of $i: Y \to C_f$ inclusion

let $i_X: X \to C_f : x \mapsto (x,1)$ be inclusion

$$\begin{tikzcd} X \arrow[rr, hook, "i_X"] \arrow[dr, "f"'] & \circ & C_f \arrow[dl, "j"] \\ & Y & \end{tikzcd}$$

under homotopy equiv j, $f \simeq i_X$

$\therefore$ since $f_*: \pi_n(X) \to \pi_n(Y)$ $\cong$ $\forall n$ we also have $(i_X)_*: \pi_n(X) \to \pi_n(C_f)$ $\cong$ $\forall n$

thus by the long exact sequence in lem 17

$$\pi_n(X) \xrightarrow{\cong} \pi_n(C_f) \to \pi_n(C_f, X) \to \pi_{n-1}(X) \xrightarrow{\cong} \pi_{n-1}(C_f)$$
$$\pi_n(C_f, X) = 0$$

$\therefore$ Th$^{\underline{m}}$ 24 $\Rightarrow$ C_f deformation retracts to X

i.e. $C_f \simeq X$ $\therefore$ $X \simeq C_f \simeq Y$ ▦

Recall for all these results we need lemma 20

to prove lemma 20 we need a technical lemma

lemma 26 (Homotopy Extension Theorem)

given • a relative CW pair (X, A)
- a map $f: X \to Y$ and
- a homotopy $H: A \times [0,1] \to Y$ of $f|_A$

then there is an extension of H to

$G: X \times [0,1] \to Y$

such that $G(x,t) = H(x,t)$ $\forall x \in A, t$ and
$G(x,0) = f(x)$

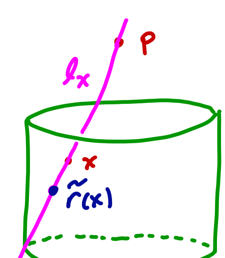
exercise: Directly prove Th$^{\underline{m}}$ 21 and 24 using Lemma 26

Proof: Main point: for any disk D^n there is a deformation retraction of $D^n \times [0,1]$ to $D^n \times \{0\} \cup \partial D^n \times [0,1]$

Pf: let $D^n \subset \mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$

so $D^n \times [0,1] \subset \mathbb{R}^{n+1}$

let $p = (0, 0, \ldots, 0, 2)$



given $x \in D^n \times [0,1]$ let l_x = line through x and p

and set $\tilde{r}(x) = l_x \cap (D^n \times \{0\} \cup \partial D^n \times [0,1])$

unique point!

clear $\tilde{r}$ is a retraction (need to check continuous exercise)
and $\tilde{r}_t = t\tilde{r} + (1-t)\,\mathrm{id}_{D^n \times [0,1]}$
is a deformation retraction

now suppose $X - A$ is just one cell D^n, $\partial D^n \subset A$

note by hypothesis we have a map

$$\bar{H} : \underbrace{(X \times \{0\}) \cup (A \times [0,1])}_{B} \to Y : \begin{cases} (x,0) \longmapsto f(x) \\ (x,t) \longmapsto H(x,t) & x \in A \end{cases}$$

now set $G : X \times [0,1] \to Y : \begin{cases} \bar{H}(x,t) & \text{if } x \in B \\ \bar{H} \circ \tilde{r}(x,t) & \text{if } x \in D^n \times [0,1] \end{cases}$

this is clearly an extension!

for a general X, we just do this cell by cell

Proof of lemma 20: recall we know A is contractible

so $\exists$ a homotopy $f : A \times [0,1] \longrightarrow A \subset X$ st.

$f_0(x) = f(x,0) = x$

$f_1(x) = f(x,1)$ is constant

note $f_0 = F_0|_A$ where $F_0 = id_X$

so HET gives a homotopy $F : X \times [0,1] \longrightarrow X$ extending f

since $F_t(A) \subset A$ for all t we get maps $\bar{F}_t : X/A \to X/A$

$$\begin{tikzcd} X \arrow[r, "F_t"] \arrow[d, "q"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{F}_t"'] & X/A \end{tikzcd}$$

also $F_1(A) = pt$ so F_1 also gives a map $h: X/A \to X$

$$\begin{tikzcd} X \arrow[r, "F_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[ur, "h"] \arrow[r, "\bar{F}_1"'] & X/A \end{tikzcd}$$

you can easily check $h \circ q = F_1$ and $q \circ h = \bar{F}_1$

but now $h \circ q = F_1 \sim F_0 = id_X$

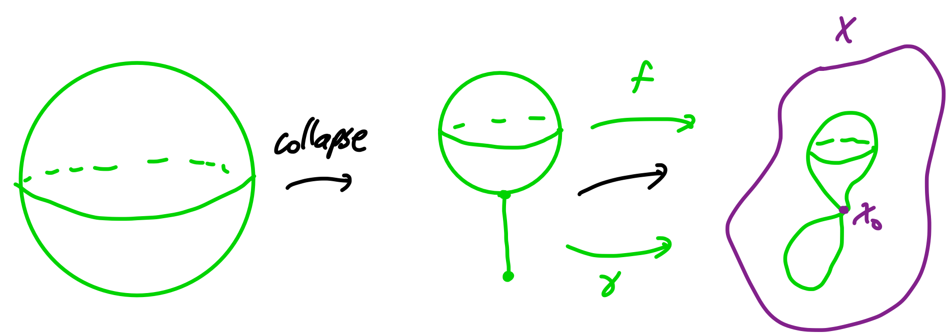
$q \circ h = \bar{F}_1 \sim \bar{F}_0 = id_{X/A}$

back to computing π_k's

recall by lemma 10 $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$

given $[\gamma] \in \pi_1(X, x_0)$ and $[f] \in \pi_n(X, x_0)$ let

$\gamma \cdot f$ be



and $[\gamma] \cdot [f] = [\gamma \cdot f]$

exercise: this makes $\pi_n(X, x_0)$ a $\pi_1(X, x_0)$-module

Thm 27:

let
- X be a topological space, $x_0 \in X$
- $f: \partial D^n \to X$ a map, $y_0 \in D^n$ and $f(y_0) = x_0$
- $\hat{X} = X \cup_f D^n = X \cup D^2/\sim$ where $y \in \partial D^n \sim f(y) \in X$
- $i: X \to \hat{X}$ the inclusion map

then $\imath_*: \pi_k(X, x_0) \to \pi_k(\hat{X}, x_0)$ is an isomorphism for $k < n-1$ and surjective for $k = n-1$ with kernel the normal subgroup generated by $[f]$ and $[\gamma]\cdot[f]$ for all $[\gamma] \in \pi_1(X, x_0)$

Proof: given $g: S^k \to \hat{X}$ consider $g^{-1}(\text{int } D^n)$

this is a smooth submfd of S^k so we can isotop g to be smooth in some nbhd of $f^{-1}(p)$ some $p \in \text{int } D^n$ and then to be transverse to p

so if $k < n$ then $f^{-1}(p) = \emptyset$ and since $D^n - p$ retracts to ∂D^n, g is isotopic to a map with image in X

i.e. $\imath_*$ is onto for $k < n$

similarly if $g_0, g_1: S^k \to \hat{X}$ are homotopic via $H: S^k \times [0,1] \to \hat{X}$ and $k < n-1$ then we can homotop H to have image in X, $\therefore$ $\imath_*$ is injective for $k < n-1$

now for $\imath_*: \pi_{n-1}(X, x_0) \to \pi_{n-1}(\hat{X}, x_0)$

clearly $[f]$ and $[\gamma]\cdot[f]$ in $\ker \imath_*$ for all $[\gamma] \in \pi_1(X, x_0)$

so we are left to see $[g] \in \ker \iota_*$

is in smallest normal subgp gen by $[\gamma]\cdot[f]$

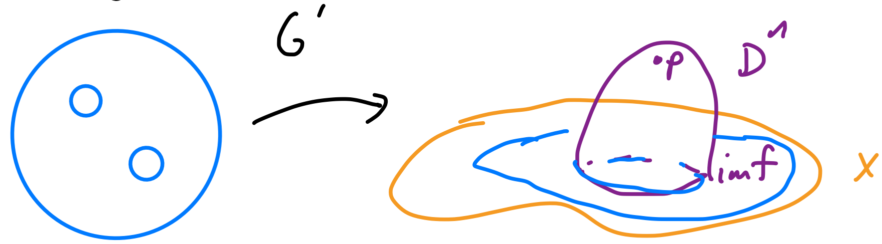
we know $\exists\, G : D^n \longrightarrow \hat{X}$ such that $G|_{\partial D^n} = g$

equivalent to g null-homotopic

as above, we can assume $G^{-1}(p) = \{p_1 \dots p_n\}$

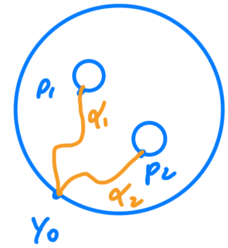
let N_i be a nbhd of p_i in D^n

as above, $G|_{D^n \setminus \cup N_i}$ is homotopic to G' with image in X and each boundary component of $(\partial(D^n \setminus \cup_{N_i}) - \partial D^n)$ has image equal to image(f)



so $\exists\, p_i \in \partial N_i$ s.t. $G'(p_i) = x_0$

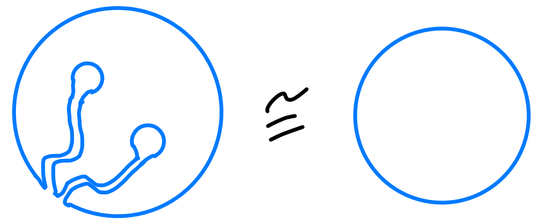
let $\alpha_i : [0,1] \longrightarrow D^n \setminus \cup N_i$ be path y_0 to p_i



claim: $[g] = \prod [G' \circ \alpha_i] \cdot G'(\partial N_1) = \prod [G' \circ \alpha_1]\cdot[f]$

indeed, note D^n maps onto $(D^n \setminus \cup N_i) \setminus (\cup \text{im}\, G' \circ \alpha_i)$

$\left((D^n \setminus \cup N_i) \setminus (\cup \text{ im } G'_{\alpha_i}) \text{ is } D^n \right)$



so $[g]\left(\prod [G'_{\alpha_i}]\cdot[f]\right)^{-1} \simeq$ constant

Th$^{\underline{m}}$ 28:

> any topological space is weakly homotopy equivalent to a CW complex

Proof: let X be a topological space

(we can assume X is path connected)

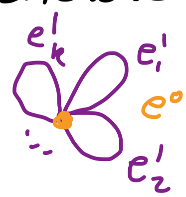
let $x_0 \in X$

Set $Y_0 = \{e^0\}$ and $f_0 : Y_0 \to X : e^0 \mapsto x_0$

f_0 induces an isomorphism on π_0

let $\alpha_1, \dots, \alpha_k : [0,1] \to X$ generate $\pi_1(X, x_0)$

Set $Y_1' = Y_0 \cup e_1^1 \cup \dots \cup e_k^1$



extend f_0 to $f_1' : Y_1' \to X$ by α_i on each e_i^1

clearly f_1' induces an isomorphism on π_n, $n<1$ and surjective for $n=1$

let $\beta_1 \dots, \beta_\ell$ generate the kernel of $(f_1')_*$ on $\pi_1(Y_1')$

so $f_1' \circ \beta_i : [0,1] \to X$ is null-homotopic

and we have disks $F_i : D^2 \to X$ s.t.
$$F_i|_{\partial D^2} = f_1' \circ \beta_i$$

let $Y_1 = Y_1' \cup \coprod_{i=1}^{\ell} \bar{e}_i^2$ where $\bar{e}_i^2$ attached via $\beta_i : \partial \bar{e}_i^2 \to Y_1'$

extend f_1' to $f_1 : Y_1 \to X$ by F_i on $\bar{e}_i^2$

exercise: $\pi_1(Y_1, e^0) = \pi_1(Y_1', e^0) / \langle \beta_1 \dots \beta_\ell \rangle$

so f_1 induces an isomorphism on π_n for $n \leq 1$

now let $\alpha_1 \dots \alpha_k : D^2 \to Y$ generate $\pi_2(X, x_0)$

Set $Y_2' = Y_1 \cup e_1^2 \cup \dots e_k^2$ where each e_i^2 attached
$= Y_1 \vee \left(\bigvee_{i=1}^{k} S_i^2 \right)$ by constant map

and extend f_1 to $f_2' : Y_2' \longrightarrow X$ by α_i on e_i^2

clearly f_2' induces an isomorphism on π_n, $n < 2$
and surjective for $n = 2$

let $\beta_1 \dots \beta_\ell$ generate the kernel of f_2' on π_2

as above $f_2' \circ \beta_i$ is trivial in $\pi_2(X, x_0)$

so $\exists$ $F_i : D^3 \longrightarrow X$ s.t. $F_i|_{\partial D^3} = \beta_i$

Set $Y_2 = Y_2' \cup \left(\coprod_{i=1}^{\ell} \bar{e}_i^3 \right)$ where $\bar{e}_i^3$ glued to Y_2'
by β_i

extend f_2' to $f_2 : Y_2 \longrightarrow X$ by F_i on $\bar{e}_i^3$

using Th$^{\underline{m}}$ 27 it is easy to see

f_2 induces an isomorphism on π_n, $n \leq 2$

continue by induction

given an element $[f] \in \pi_k(X, x_0)$

$$f : S^k \longrightarrow X$$

we can define $h_k([f]) = f_*(1) \in H_k(X)$

where 1 is a generator of $\pi_k(S^k) \cong \mathbb{Z}$

this gives a well-defined map check this if not obvious!

$$\boxed{h_k : \pi_k(X, x_0) \longrightarrow H_k(X)}$$

called the Hurewicz map

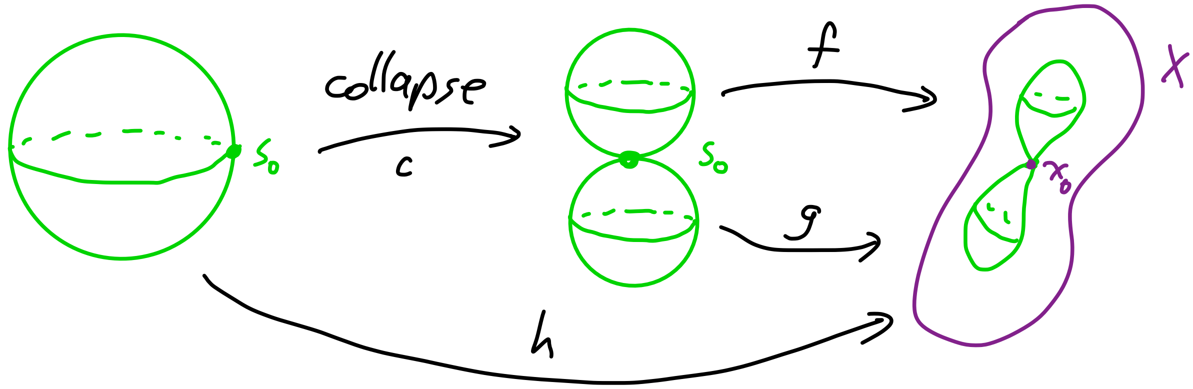
lemma 29:

$$\boxed{h_k \text{ is a homomorphism}}$$

Proof: $[f], [g] \in \pi_k(X, x_0)$

so $f, g : S^k \longrightarrow X$

$h \in [f] \cdot [g]$ is given by



$$c_*: H^k(S^k) \longrightarrow H^k(S^k \vee S^k) \cong H^k(S^k) \oplus H^k(S^k)$$
$$1 \longmapsto (1,1)$$

so $h_*(1) = (f_*, g_*) \circ c_*(1) = (f_*, g_*)(1,1) = f_*(1) + g_*(1)$

so $h_k([f]\cdot[g]) = h_k([f]) + h_k([g])$

Th$^{\underline{m}}$ 30 (Hurewicz Th$^{\underline{m}}$):

> if X is path connected, then
>
> for $n>1$, if $\pi_k(X)=0$ for $k<n$, then
>
> $$h_n: \pi_n(X) \to H_n(X) \text{ is an isomorphism}$$
>
> for $n=1$ $\ker(h_1) = [\pi_1(X), \pi_1(X)]$

Remark: 1) th$^{\underline{m}}$ says if $n>2$ is the first n s.t. $\pi_n(X) \neq 0$ then $H_k(X) = 0$ $\forall$ $k<0$ and $\pi_n(X) \cong H_n(X)$

2) similar th$^{\underline{m}}$ for $\pi_k(X,A)$ if A is simply connected

lemma 31:

$$h_k : \pi_k(S^k) \to H_k(S^k)$$
is an isomorphism

we prove this later

note: we finally know that $\pi_k(S^k) \cong \mathbb{Z}$!
and if $g : S^k \to S^k$ then $[g] \in \pi_k(S^k)$ is degree g

Corollary 32:

$$h_k : \pi_k\left(\bigvee_n S^k\right) \longrightarrow H_k\left(\bigvee_n S^k\right) \cong \bigoplus_n \mathbb{Z}$$
is an isomorphism

Proof: check $\pi_k\left(\bigvee_n S^k\right) = \bigoplus_n \pi_k(S^k)$

and $H_k(X \vee Y) \cong H_k(X) \oplus H_k(Y)$

if the wedge point $x \in X$ and $y \in Y$

satisfy $(X, x), (Y, y)$ are NDR pairs

exercise: prove this but be careful

it is not true that $\pi_k(X \vee Y) \cong \pi_k(X) \oplus \pi_k(Y)$

(for example consider $\pi_2(S^1 \vee S^2)$

its universal cover is $\simeq \bigvee_\infty S^2$) 

Proof of Thm 30 (Hurewicz Thm):

We prove the theorem for CW complexes

(true for general spaces, but harder)

let X be a CW complex s.t.

$$\pi_k(X) = 0 \quad \forall\, k < n$$

Cor 23 says $\tilde{H}_k(X) = 0 \quad \forall\, k < n$

Th$^{\underline{m}}$ 21 says we can assum $X^{(n-1)} = \{e^\circ\}$

so $X^{(n)} = \bigvee_{i \in J} S^n_i$ one for each n-cell e^n_i

$\therefore$ the cellular chain groups are

$$C_n(X) = \bigoplus_{i \in J} \mathbb{Z} \qquad e^n_i \text{ are generators}$$

$\partial^{cw} e^n_i = 0$ since $C_{n-1}(X) = \{0\}$

so $H_n(X) = C_n(X) / im(\partial^{cw} : C_{n+1}(X) \to C_n(X))$

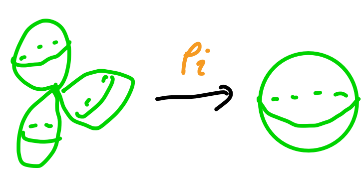
given $\beta_j : D^{n+1} \to X^{(n)}$ an $(n+1)$-cell,

recall $\partial^{cw} \beta_j$ is computed as follows

$\forall i \in J$ consider

$$\partial D^{n+1} = S^n \xrightarrow{\beta_j} X^{(n)} \to X^{(n)}/X^{(n-1)} \xrightarrow{p_i} S^n$$

p_i = project to $i^{\underline{th}}$ n-cell



$$\partial^{cw} \beta_j = \sum_{i \in J} deg(p_i \circ \beta_j)\, e^n_i$$

Th$^{\underline{m}}$ 27 says $\pi_n(X, e^\circ) = \langle e^n_i \mid [\beta_j] \rangle$

now $h_n : \pi_n(X, e^\circ) \longrightarrow H_n(X)$

$$[e_i^n] \longrightarrow (e_i^n)_*(1) = [e_i^n]$$

so h_k sends generators to generators — see note after lemma 31

and $h_n(\beta_j) = h_n\left(\prod [e_i^n]^{\deg \rho_i \circ \beta_j}\right)$

$$= \sum \deg(\rho_i \circ \beta_j)\, h_n([e_i^n])$$

$$= \sum \deg(\rho_i \circ \beta_j)[e_i^n]$$

so h_k sends relations to relations

$\therefore$ h_k an isomorphism ✓

for $n=1$, since $H_1(X)$ is abelian we know $[\pi_1(X), \pi_1(X)] \subset \ker h_1$ $\therefore$ h_1 induces

$$\bar{h}_1 : \pi_1(X)/[\pi_1(X), \pi_1(X)] \longrightarrow H_1(X)$$

as above $\bar{h}_1$ takes generators to generators and relations to relations ▦

<u>Proof of Lemma 31:</u> by 29 we know h_k a homomorphism

let $f: S^k \to S^k$ be the identity map

clearly $h_k([f]) = 1 \in H_k(S^k)$

so h_k is onto

for injectivity we note $h_k([f]) = f_*(1)$

but the definition of $\deg(f)$ is $f_*(1)$

so $h_k([f]) = \deg(f)$ for $f: S^k \to S^k$

now suppose $h_k([f]) = 0$ i.e. $\deg f = 0$

we need to show f is homotopic to a constant map

we do this by induction on n

recall, in Alg Top I we compute $H_1(S^1) \cong \mathbb{Z}$ and $h_1 : \pi_1(S^1) \to H_1(S^1)$ is an isomorphism

so base case done

now given $f : S^k \to S^k$ s.t. $\deg f = 0$

we can homotop f untill it is smooth

and take a regular value p of f

so $f^{-1}(p)$ is a finite number of pts $x_1, \dots, x_\ell$

at each x_i, df_{x_i} either preserves or reverses the orientation on S^k

set $s(x_i) = \begin{cases} +1 & df_{x_i} \text{ preserves or } " \\ -1 & df_{x_i} \text{ reverses or } " \end{cases}$

from Alg Top I we know

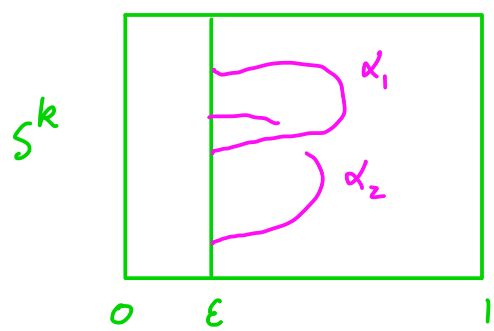
$$\deg(f) = \sum_{i=1}^{\ell} s(x_i)$$

so we can pair the points

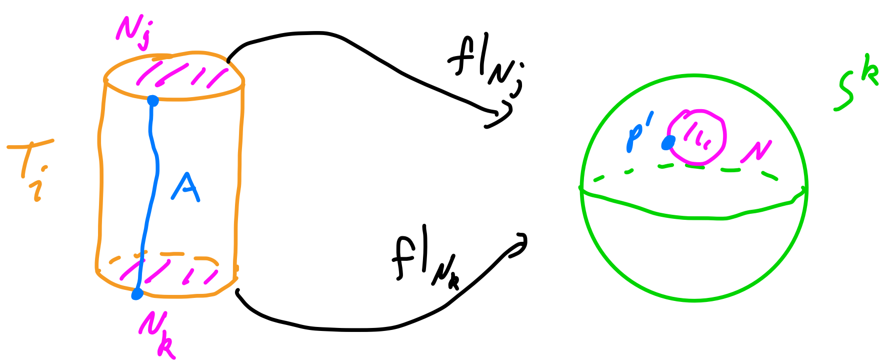
from Diff. Top. we know that since df_{x_i} is an isomorphism, that f is a diffeomorphism

from a nbhd of x_i to a nbhd of p
so we can choose a small enough nbhd N
of p s.t. $f^{-1}(N) = \cup N_i$ where N_i are
nbhds of x_i and they are disjoint

let α_i be disjoint arcs in $S^k \times [\varepsilon, 1]$ s.t.
$\partial \alpha_i$ is a pair of x_j in $S^k \times \{\varepsilon\}$
with opposite sign and
$\operatorname{int} \alpha_i \subset S^k \times (\varepsilon, 1)$
and all x_j are end pts of some α_r



let $T_i = \alpha_i \times D^k$ be a nbhd of α_i
s.t. $T_i \cap (S^k \times \{\varepsilon\}) =$ the nbhds N_j of $\partial \alpha_i$



let $p' \in \partial N$ and $p_j \in \partial N_j$ s.t. $f(p_j) = p'$

Consider $S^{k-1} \times [0,1]$ ⟵ part of ∂T_i

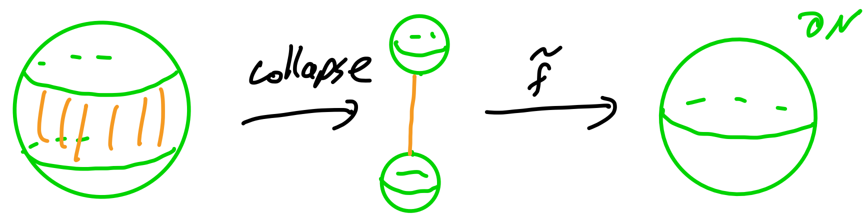
let A be an arc in $S^{k-1} \times [0,1]$
from p_j to p_k

use f to define a function

$$(S^{k-1} \times \{0,1\}) \cup A \xrightarrow{\tilde{f}} \partial N = S^{k-1}$$

constant $= p'$ on A

this gives a map $\bar{f} : S^{k-1} \to S^{k-1}$



since $\tilde{f}|_{S^{k-1} \times \{0\}}$ is a orientation reversing diffeo and
$\tilde{f}|_{S^{k-1} \times \{1\}}$ " " preserving diffeo

we know $\deg \bar{f} = 0$

so by induction $\bar{f}$ is null-homotopic

$\therefore$ we can extend $\bar{f}$ to a map $\bar{F} : D^k \to \partial N$
with $\bar{F}|_{\partial D^k} = \bar{f}$

we can use $\bar{F}$ to get a map from

$$S^{k-1} \times [0,1] \xrightarrow{\bar{F}} \partial N$$

(note $(S^{k-1} \times [0,1]) \setminus A = D^k$)

now $f|_{N_j}$ and $\bar{F}$ give a map

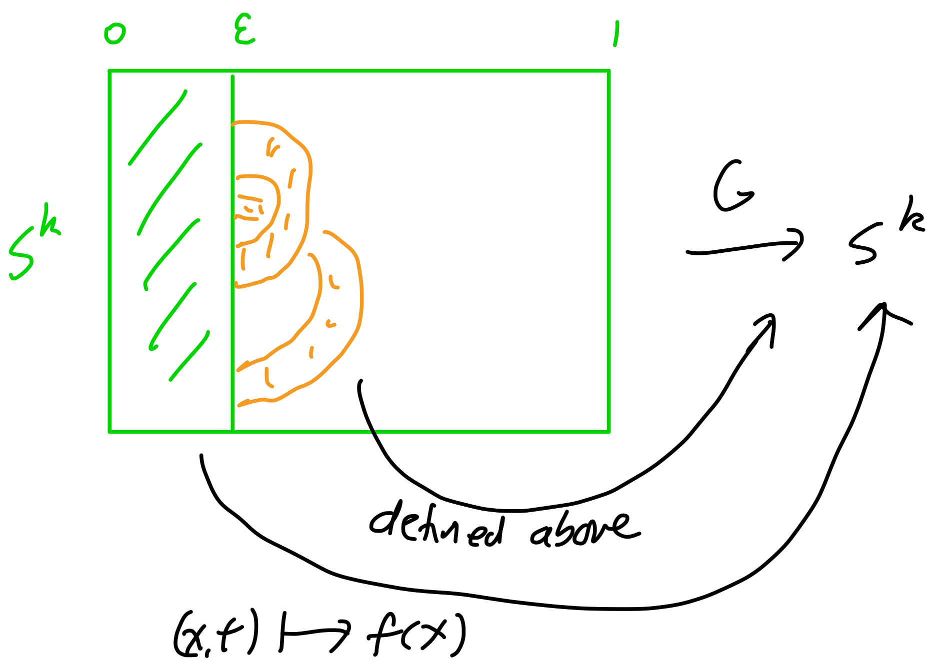
from $\underset{S^{k-1}}{\underset{\shortparallel}{\partial T_i}} \longrightarrow N = D^k$

but any map from $\partial D^k \to D^k$ can be extended to a map $D^k \to D^k$ (since it is null homotopic)

note this map takes $\partial D^k \times [0,1]$ to ∂N

we now have a map defined on

$$M = S^k \times [0,\varepsilon] \cup (\bigcup T_i) \xrightarrow{G} S^k$$



$\partial M = (S^k \times \{0\}) \cup Y$ and $G(Y) \subset S^k - \{p\}$

since $S^k - \{p\} \simeq pt$ $G|_Y$ is homotopic to a map constantly p"

let $Y \times [0,1]$ be a nbhd of Y in $(S^k \times [0,1]) - M$

can extend G over $Y \times [0,1]$ by the above homotopy

and then over rest of $S^k \times [0,1]$ by sending everything to p''

this gives a homotopy of f to a constant map $\blacksquare$

lemma 33:

If $\pi_1(X) = 1$ and $f: X \to Y$ induces an isom

$$H_k(X) \longrightarrow H_k(Y) \quad \forall\ k \leq n$$

then it induces an isom

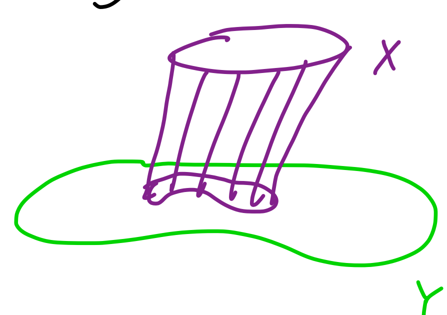
$$\pi_k(X) \to \pi_k(Y) \quad \forall k < n$$

and is onto for $k = n$

Remark: also true with π_k, H_k interchanged.

Proof: let C_f be the mapping cylinder of f

$Y \hookrightarrow C_f$ is a retract of C_f so



$$\pi_k(C_f) \cong \pi_k(Y) \text{ and } H_k(C_f) \cong \pi_k(Y) \quad \forall k$$

$\exists$ inclusion $i_X : X \to C_f$ s.t.

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & C_f \arrow[d, "j"] \\ & Y \end{tikzcd}$$

so $i_X \simeq f$ via j

exercise: C_f and X simply connected $\Rightarrow \pi_1(C_f, X) = 0$

$$\begin{tikzcd} H_i(X) \arrow[r, "f_*"] & H_i(Y) \arrow[r] & H_i(C_f, X) \arrow[r] & H_{i-1}(X) \arrow[r, "f_*"] & H_{i-1}(Y) \\ \pi_i(X) \arrow[r, "f_*"] \arrow[u, "h_i"] & \pi_i(Y) \arrow[r] \arrow[u, "h_i"] & \pi_i(C_f, X) \arrow[r] \arrow[u, "h_i"] & \pi_{i-1}(X) \arrow[r, "f_*"] \arrow[u, "h_{i-1}"] & \pi_{i-1}(Y) \arrow[u, "h_{i-1}"] \end{tikzcd}$$

exercise: diagram commutes (only nontrivial one to check is 3rd square)

f_* an isomorphism for top row with $i \le n$

$\Rightarrow H_i(C_f, X) = 0$ for $i \le n$

relative Hurewicz $\Rightarrow \pi_i(C_f, X) = 0$ for $i \le n$

$\Rightarrow f_* : \pi_i(X) \to \pi_i(Y)$ an isom $i < n$ and surjective for $i = n$

Th^m 34:

$\boxed{\text{If } X, Y \text{ are simply connected CW complexes and } f: X \to Y \text{ induces an isomorphism } f_* : H_k(X) \to H_k(Y)\ \forall k \text{ then } f \text{ is a homotopy equivalence}}$

Proof: lemma 33 says f induces an isomorphism on π_k for all k
so Whitehead's Th$^{\underline{m}}$ (Th$^{\underline{m}}$ 25) says f is a homotopy equivalence $\boxplus$

given a group π and a positive integer n
s.t. π is abelian if $n>1$
Then a topological space X is an Eilenberg-MacLane space of type (π, n) or simply a $K(\pi, n)$ if

$$\pi_k(X) = \begin{cases} 0 & k \neq n \\ \pi & k = n \end{cases}$$

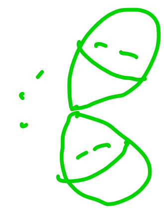
example: S^1 is a $K(\mathbb{Z}, 1)$
indeed $\pi_1(S^1) = \mathbb{Z}$ and by Th$^{\underline{m}}$ 18
$\pi_k(S^1) \cong \pi_k(\mathbb{R}) = 0 \quad \forall k>1$

Th$^{\underline{m}}$ 35

given any group and integer as above
$\exists$ a CW complex that is a $K(\pi, n)$
and it is unique upto homotopy

Proof: assume $n>1$ (exercise: do $n=1$ case) let $\{\alpha_i\}_{i \in I}$ be a generating set for π

set $\hat{X} = $ 0-cell $\cup \{e_i^n\}_{i \in I}$



$$= \bigvee_{i \in I} S^n$$

$\pi_k(\hat{X}) = 0 \quad \forall k < n$ and

$\pi_n(\hat{X}) \cong H_n(\hat{X}) \cong \bigoplus_{i \in I} \mathbb{Z}\langle e_i^n \rangle$ by Hurewicz

let $\{r_j\}_{j \in J}$ be relations for π

for each r_j, $\exists$ a map $f_j : S^n \to \hat{X}$
s.t. $r_j = [f_j] \in \pi_n(\hat{X})$ } exercise if not clear

since $\hat{X}$ is simply connected, Th^m 27 says
attaching e_j^{n+1} to $\hat{X}$ with $f_j : \partial e_j^{n+1} \to \hat{X}$
will add the relation r_j to π_n
(also all π_k, $k<n$, are unaffected)

so if $\bar{X} = \hat{X} \cup \{e_j^{n+1}\}$ using the f_j

then $\pi_k(\bar{X}) = \begin{cases} 0 & k < n \\ \pi & k = n \end{cases}$

now $\pi_{n+1}(\bar{X})$ is generated by so elements

$$g_i : S^{n+1} \longrightarrow \bar{X}$$

add n+2 cells to $\bar{X}$ using g_i to get $\tilde{X}$

now $\pi_k(\tilde{X}) = \begin{cases} 0 & k<n,\ k = n+1 \\ \pi & k = n \end{cases}$

inductively kill π_k for $k > n$ to get $K(\pi, n)$

to show uniqueness up to htpy equivalence
let X, Y be two CW $K(\pi, n)$s

if we can construct a map $f: X \to Y$
inducing an isomorphism on all π_k
then we are done by Whitehead's Th$^{\underline{m}}$
the construction of f is exactly like
in the proof of following Th$^{\underline{m}}$

Th$^{\underline{m}}$36:

If X, Y are connected CW complexes and
X is a $K(\pi, 1)$
then $\exists$ a one-to-one correspondence

$$[(Y, y_0), (X, x_0)]_0 \longleftrightarrow \mathrm{Hom}(\pi_1(Y, y_0), \pi_1(X, x_0))$$

a connected space with $\pi_k = 0 \ \forall k \geq 2$
is called asphenical

Proof: we assume $X^{(0)} = \{x_0\}, Y^{(0)} = \{y_0\}$
clearly if $[f] \in [Y, X]_0$ then
$f_*: \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is
a homomorphism

Claim: this map is onto

indeed, given $h: \pi_1(Y) \to \pi_1(X)$

we will inductively build a map

$$f: Y \to X \quad \text{s.t. } f_* = h$$

on the 0-skeleton map is obvious

$$f(y_0) = x_0$$

each 1-cell e^1 in Y is a loop, $[e^1] \in \pi_1(Y)$

so $h([e^1]) \in \pi_1(X)$

let $\gamma \in h([e^1])$ so $\gamma: [0,1] \to X$

define f on e^1 by γ

this extends f to $Y^{(1)} \to X$

given a 2-cell e^2 in $Y^{(2)}$

∂e^2 is a loop η in $Y^{(1)}$

and $[\eta] = 0 \in \pi_1(Y)$

since bounds e^2

so $[f(\eta)] = h([\eta]) = 0$ in $\pi_1(X)$

$\therefore$ $f(\eta)$ bounds a disk $F: D^2 \to X$ in X

use F to extend f over e^2

this extends f over $Y^{(2)} \to X$

now inductively assume f is

defined $Y^{(k)} \to X$

assume e^{k+1} is a $(k+1)$ cell in $Y^{(k+1)}$

$\partial e^{k+1} \subset Y^{(k)}$

so $f(\partial e^{k+1})$ is a k-sphere in X

since $\pi_k(X) = 0$ $(k > 1)$ we know

this extends to a $(k+1)$-disk

i.e. $\exists F: D^{k+1} \to X$ s.t.

$F(\partial D^{k+1}) = f(\partial e^{k+1})$

use F to extend f over e^{k+1}

$\therefore$ we can extend f over $Y^{(k+1)} \to X$

we now have $f: Y \to X$ and f_* and

h act the same on the generators

of $\pi_1(Y)$ (i.e. on loops in $Y^{(1)}$)

$\therefore f_* = h$ and our map onto ✓

Claim: map is injective

Suppose $f, g: Y \to X$ induce same

map $\pi_1(Y) \to \pi_1(X)$

consider the CW str on $Y \times [0,1]$

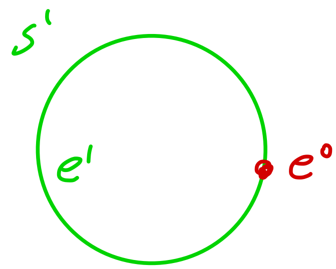
for each i-cell e^i of Y get

2 i-cells of $Y \times [0,1]$

$e^i \times \{0\}$ and $e^i \times \{1\}$ and

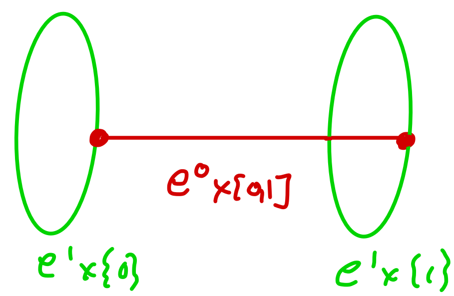
1 $(i+1)$-cell " "

$\tilde{e}^i = e^i \times [0,1]$ attached in "obvious way

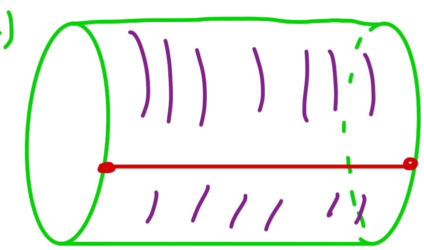


$(S^1 \times [0,1])^{(0)}$

$(S^1 \times [0,1])^{(1)}$



$(S^1 \times [0,1])^{(2)}$



$\tilde{e}^1 = e^1 \times [0,1]$

define $H : Y \times [0,1] \longrightarrow X$ by

f on $Y \times \{0\}$

g on $Y \times \{1\}$

x_0 on $e^0 \times [0,1]$

now inductively define over $\tilde{e}^1$

for $\tilde{e}^1$ in $(Y\times\{0,1])^{(2)}$ note

← go backwards

$$\partial\tilde{e}^1 = \overline{(e^1\times\{0\})} * (e^0\times[0,1]) * (e^1\times\{1\}) * (e^1\times\{1\})$$

so $H(\partial\tilde{e}^1) = \overline{f(e^1)} * x_0 * g(e^1) * x_0$

$\simeq \overline{f(e^1)} * g(e^1)$

and this is 0 in $\pi_1(X)$

so $\exists$ a disk D^2 in X with boundary $H(\partial\tilde{e}^1)$

extend H over $\tilde{e}^1$ using D^2

so H defined on $(Y\times[0,1])^{(2)}$

can extend H over higher skeleta just like above